

Gravity Amplitudes from n -Space

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Abstract

We identify a hidden $GL(n, \mathbb{C})$ symmetry of the tree level n -point MHV gravity amplitude. Representations of this symmetry reside in an auxiliary n -space whose indices are external particle labels. Spinor helicity variables transform non-linearly under $GL(n, \mathbb{C})$, but linearly under its notable subgroups, the little group and the permutation group S_n . Using $GL(n, \mathbb{C})$ covariant variables, we present a new and simple formula for the MHV amplitude which can be derived solely from geometric constraints. This expression carries a huge intrinsic redundancy which can be parameterized by a pair of reference 3-planes in n -space. Fixing this redundancy in a particular way, we reproduce the S_{n-3} symmetric form of the MHV amplitude of [1], which is in turn equivalent to the S_{n-2} symmetric form of [2] as a consequence of the matrix tree theorem. The redundancy of the amplitude can also be fixed in a way that fully preserves S_n , yielding new and manifestly S_n symmetric forms of the MHV amplitude. Remarkably, these expressions need not be manifestly homogeneous in spinorial weight or mass dimension. We comment on possible extensions to N^{k-2} MHV amplitudes and speculate on the deeper origins of $GL(n, \mathbb{C})$.

1 Geometry of n -Space

In this paper we argue that the tree level n -point MHV gravity amplitude possesses a hidden $GL(n, \mathbb{C})$ symmetry. This symmetry acts on an auxiliary n -space indexed by the labels of external particles, $a \in \{1, 2, \dots, n\}$. We will show that geometric constraints in n -space are sufficient to derive a new and simple expression for the MHV gravity amplitude in terms of $GL(n, \mathbb{C})$ covariant objects. By virtue of $GL(n, \mathbb{C})$ this formula is manifestly symmetric under the permutation group S_n . Our formula contains a large redundancy which originates from the geometric constraints. Fixing this redundancy yields the expressions for the MHV gravity amplitude in the literature, as well as new ones.

Consider a general n -point amplitude, written as a function of spinor helicity variables, $\lambda_{a\alpha}$ and $\tilde{\lambda}^{a\dot{\alpha}}$. What is the action of $GL(n, \mathbb{C})$ on these kinematic variables? Naively, it is natural to define the spinors $\lambda_{a\alpha}$ and $\tilde{\lambda}^{a\dot{\alpha}}$ as 2-planes which transform as fundamental and anti-fundamental representations of $GL(n, \mathbb{C})$, so

$$\lambda_{a\alpha} \rightarrow \sum_b \lambda_{b\alpha} G^b_a \quad , \quad \tilde{\lambda}^{a\dot{\alpha}} \rightarrow \sum_b \tilde{\lambda}^{b\dot{\alpha}} G_b^{-1a}. \quad (1)$$

Indeed, for this choice the condition of momentum conservation

$$\sum_a \tilde{\lambda}^{a\dot{\alpha}} \lambda_{a\alpha} = 0, \quad (2)$$

is manifestly $GL(n, \mathbb{C})$ symmetric and implies that these 2-planes are mutually orthogonal. As such, λ_a and $\tilde{\lambda}^a$ provide a natural linear representation of $GL(n, \mathbb{C})$.

However, just because $GL(n, \mathbb{C})$ acts on the particle label a does not imply that $GL(n, \mathbb{C})$ must act linearly on λ_a and $\tilde{\lambda}^a$. These representations can, in principle, be non-linear functions of λ_a and $\tilde{\lambda}^a$. In the present work we argue that for MHV gravity amplitudes the correct variables are instead

$$\begin{aligned} v_a^i &= \frac{i}{2} \sum_{\alpha, \beta} \lambda_{a\alpha} \sigma^{i\alpha\beta} \lambda_{a\beta} \quad , \quad i \in \{1, 2, 3\} \\ \phi^{ab} &= \frac{[ab]}{\langle ab \rangle} \quad , \quad a \neq b. \end{aligned} \quad (3)$$

Here v_a^i is a symmetric spinor product transforming as a $(1, 0)$ representation of the Lorentz group, and the diagonal elements ϕ^{aa} are as of yet undefined. The variable ϕ^{ab} was defined in the recent remarkable papers of Hodges on the MHV amplitude [1, 3]. Our claim is that v_a^i and ϕ^{ab} furnish linear representations of $GL(n, \mathbb{C})$, so

$$\begin{aligned} v_a^i &\rightarrow \sum_b v_b^i G^b_a \\ \phi^{ab} &\rightarrow \sum_{c,d} \phi^{cd} G_c^{-1a} G_d^{-1b}. \end{aligned} \quad (4)$$

Note the placement of raised and lowered indices—under $GL(n, \mathbb{C})$, the 3-plane v_a^i transforms as a fundamental and the tensor ϕ^{ab} transforms as a symmetric anti-bi-fundamental. Moreover, Eq. (3) and Eq. (4) imply that λ_a and $\tilde{\lambda}^a$ transform non-linearly under $GL(n, \mathbb{C})$.

From v_a^i we can now construct the Lorentz invariant, $GL(n, \mathbb{C})$ covariant tensors

$$\begin{aligned} v_{ab} &= \sum_i v_a^i v_b^i = \frac{1}{2} \langle ab \rangle^2 \\ v_{abc} &= \sum_{i,j,k} \epsilon^{ijk} v_a^i v_b^j v_c^k = \frac{1}{2} \langle ab \rangle \langle bc \rangle \langle ca \rangle. \end{aligned} \quad (5)$$

Obviously, additional tensors can be constructed from higher order products, but only the quantities above will be important for the MHV gravity amplitude.

Lastly, let us consider the little group and the permutation group S_n , which are subgroups of $GL(n, \mathbb{C})$. A little group transformation sends $\lambda_a \rightarrow w_a \lambda_a$ and $\tilde{\lambda}^a \rightarrow w_a^{-1} \tilde{\lambda}^a$, corresponding to the diagonal subgroup of $GL(n, \mathbb{C})$ defined by

$$G_a^b = \delta_a^b w_a^2, \quad (6)$$

in Eq. (4). Likewise, the action of S_n is $\lambda_a \rightarrow \lambda_{\sigma(a)}$ and $\tilde{\lambda}^a \rightarrow \tilde{\lambda}^{\sigma(a)}$, corresponding to

$$G_a^b = \delta_{\sigma(a)}^b. \quad (7)$$

Thus, even though the $GL(n, \mathbb{C})$ covariant variables ϕ^{ab} and v_a^i are non-linear functions of λ_a and $\tilde{\lambda}^a$, this is consistent with the fact that these spinors must transform linearly under the little group and S_n .

2 Amplitudes from Geometry

We assume that the MHV gravity amplitude is a function of ϕ^{ab} and v_a alone, so in terms of on-shell $\mathcal{N} = 8$ superspace, the amplitude is

$$\mathcal{M}_n^{\text{MHV}} = \delta^{2\mathcal{N}} \left(\sum_a \lambda_a \tilde{\eta}_a \right) \hat{\mathcal{M}}_n^{\text{MHV}}(\phi^{ab}, v_a). \quad (8)$$

Furthermore, we assume the crucial projection relation,

$$\sum_b \phi^{ab} v_b^i = 0, \quad (9)$$

whose importance was rightly emphasized in [1]. As we will see, the underlying $GL(n, \mathbb{C})$ symmetry together with the geometric relation in Eq. (9) entirely fix the structure of the MHV amplitude!

It is sometimes convenient to re-express the projection condition in component form

$$\sum_b \phi^{ab} |b\rangle |b\rangle = 0. \quad (10)$$

By dotting the left hand side with a symmetric product of reference spinors, $\langle x| \langle y| + \langle y| \langle x|$, we can solve for the diagonal components ϕ^{aa} yielding

$$\phi^{aa} = - \sum_{b \neq a} \frac{[ab] \langle bx \rangle \langle by \rangle}{\langle ab \rangle \langle ax \rangle \langle ay \rangle}, \quad (11)$$

precisely as defined in [1]. Of course, the explicit dependence on $|x\rangle$ and $|y\rangle$ always cancels, since these spinors are only here to provide a component form expression of the geometric constraint in Eq. (9).

What is $\hat{\mathcal{M}}_n^{\text{MHV}}$? Certainly, the amplitude should depend on ϕ^{ab} and v_a in such a way that leaves no free a indices—otherwise the MHV amplitude would not preserve $GL(n, \mathbb{C})$. The first obvious choice is $\sum_{a,b,i} \phi^{ab} v_a^i v_b^i$, but this vanishes due to the projection condition in Eq. (9). Another natural possibility is $\det(\phi^{ab})$, but this also vanishes because Eq. (9) implies that $\text{rank}(\phi^{ab}) = n - 3$. Nonetheless, there is an alternative invariant quantity which is suitable: the pseudo-determinant, defined as the product of all non-zero eigenvalues. For a general matrix M , the pseudo-determinant is formally defined as

$$\text{pdet}(M) \equiv \lim_{\epsilon \rightarrow 0} \epsilon^{-\text{null}(M)} \det(M + \epsilon \mathbb{1}), \quad (12)$$

where $\text{null}(M)$ is the nullity of M . Unfortunately, $\text{pdet}(\phi^{ab})$ yields an expression which does not have uniform little group weight, since ϕ^{ab} is covariant while the identity matrix is invariant. To get around this we construct a vierbein defined by the geometric relations

$$\begin{aligned} \sum_a v_a^i e^a_\alpha &= \delta^i_\alpha, \quad \alpha \in \{1, 2, \dots, n\}, \quad i \in \{1, 2, 3\} \\ \sum_a e^\alpha_a e^a_\beta &= \delta^\alpha_\beta. \end{aligned} \quad (13)$$

The constraint in Eq. (13) obviously does not fix the vierbein e^a_α or inverse vierbein e^α_a uniquely, since v_a^i is a 3-plane. Specifically, the definition in Eq. (13) carries an intrinsic redundancy parameterized by the $(n - 3)$ -plane orthogonal to v_a^i . This redundancy in the vierbein will later manifest itself in the MHV gravity amplitude.

Introducing “left” and “right” vierbeins which separately satisfy Eq. (13), we can transform ϕ^{ab} into an orthonormal basis,

$$\phi^{ab} \rightarrow \sum_{a,b} \phi^{ab} e^\alpha_{La} e^\beta_{Rb}, \quad (14)$$

whose pdet can be taken. The natural invariant that can be constructed in the orthonormal basis is

$$\det(e^a_{L\alpha}) \det(e^a_{R\alpha}) \text{pdet} \left(\sum_{a,b} \phi^{ab} e^\alpha_{La} e^\beta_{Rb} \right), \quad (15)$$

where the \det prefactors have been included so that this object has the little group weight of an MHV gravity amplitude. Note that unlike \det , pdet does not, in general, distribute multiplicatively over products, so the vierbeins inside and outside of the pdet cannot be canceled against each other.

The expression in Eq. (15) can be recast in slightly more palatable form by reshuffling the vierbein fields and rewriting the pdet . In particular,

$$\begin{aligned}
\text{pdet} \left(\sum_{a,b} \phi^{ab} e_{La}^\alpha e_{Rb}^\beta \right) &= \lim_{\epsilon \rightarrow 0} \epsilon^{-3} \det \left(\sum_{a,b} \phi^{ab} e_{La}^\alpha e_{Rb}^\beta + \epsilon \mathbb{1}^{\alpha\beta} \right) \\
&= \det \left(\sum_{a,b} \phi^{ab} e_{La}^\alpha e_{Rb}^\beta + \sum_i \delta^{\alpha i} \delta^{\beta i} \right) \\
&= \det(e_{L\alpha}^a)^{-1} \det(e_{R\alpha}^a)^{-1} \det \left(\phi^{ab} + \sum_{\alpha,\beta,i} e_{L\alpha}^a e_{R\beta}^b \delta^{\alpha i} \delta^{\beta i} \right). \quad (16)
\end{aligned}$$

Note that in going from the first to the second lines, it was crucial that ϕ^{ab} satisfy the projection relation in Eq. (9) and the vierbeins satisfy the defining relation in Eq. (13). Given these geometric constraints, the second line follows from the first line as a matter of linear algebra—it has nothing to do with the kinematic information implicitly contained in the auxiliary variables.

Combining the above expression with Eq. (15) yields our final result: a new form of the MHV gravity amplitude which is $GL(n, \mathbb{C})$ symmetric:

$$\hat{\mathcal{M}}_n^{\text{MHV}} = \frac{1}{4} \det \left(\phi^{ab} + \sum_{\alpha,\beta,i} e_{L\alpha}^a e_{R\beta}^b \delta^{\alpha i} \delta^{\beta i} \right). \quad (17)$$

Here the constant normalization of the amplitude is not fixed by $GL(n, \mathbb{C})$, so we take it from the known expression. We emphasize that *any* choice of the vierbeins $e_{L\alpha}^a$ and $e_{R\alpha}^a$ which satisfies Eq. (13) will produce the correct amplitude. However, as noted earlier, Eq. (13) carries a huge ambiguity in the definition of $e_{L\alpha}^a$ and $e_{R\alpha}^a$. This redundancy can be parameterized as a pair of $(n-3)$ -planes. In an abuse of nomenclature, we will refer to this redundancy as a “gauge freedom” which can be “gauge fixed” by an appropriate choice of reference $(n-3)$ -planes. As we will see, gauge fixing the redundancy reproduces the known MHV formulas in the literature.

Under the $GL(n, \mathbb{C})$ transformation in Eq. (4), the MHV amplitude transforms as

$$\hat{\mathcal{M}}_n^{\text{MHV}} \rightarrow \det(G_b^a)^{-2} \hat{\mathcal{M}}_n^{\text{MHV}}. \quad (18)$$

Restricting G_b^a to little group transformations, we see that Eq. (18) implies that Eq. (17) has the proper spinorial weight. The transformation law in Eq. (18) strongly suggests that the MHV amplitude is secretly a density or integral measure over $GL(n, \mathbb{C})$ covariant variables.

Eq. (17) can be expressed in an alternative but equivalent form which is more simple for computations. In particular, note that

$$\sum_{\alpha,\beta,i} e_{L\alpha}^a e_{R\beta}^b \delta^{\alpha i} \delta^{\beta i} = \sum_i e_L^{ai} e_R^{bi}, \quad (19)$$

is simply the outer product of a pair of 3-planes, which is a rank 3 matrix. However, this is not any old rank 3 matrix, since it is constrained by the definition of the vierbeins in Eq. (13). Nonetheless, Eq. (17) can be reassembled in terms of an arbitrary rank 3 matrix, at the cost of some Jacobian factors which depend on v_a^i . Without loss of generality, we can parameterize this arbitrary rank 3 matrix with $\sum_i L^{ai} R^{bi}$, where L^{ai} and R^{bi} are arbitrary 3-planes. This effectively dualizes the pair of $(n-3)$ -planes which parameterize the gauge freedom into a pair of 3-planes. Using linear algebraic identities, we find that Eq. (17) has a second equivalent form,

$$\hat{\mathcal{M}}_n^{\text{MHV}} = \frac{\det \left(\phi^{ab} + \sum_i L^{ai} R^{bi} \right)}{\left(\sum_{a,b,c} L^{abc} v_{abc} / 3 \right) \left(\sum_{d,e,f} R^{def} v_{def} / 3 \right)}, \quad (20)$$

where the denominators are simple Jacobian factors, and in analogy with Eq. (5) we have defined

$$\begin{aligned} L^{abc} &= \sum_{i,j,k} \epsilon^{ijk} L^{ai} L^{bj} L^{ck} \\ R^{abc} &= \sum_{i,j,k} \epsilon^{ijk} R^{ai} R^{bj} R^{ck}. \end{aligned} \quad (21)$$

Eq. (20) holds for *any* choice of the reference 3-planes, L^{ai} and R^{bi} . Note that while Eq. (20) is manifestly S_n symmetric, it is not simply a summation over permutations of an existing form of the MHV gravity amplitude.

3 Equivalent Representations from Gauge Fixing

Our expression for the MHV amplitude in Eq. (20) has a huge redundancy parameterized by the reference 3-planes, L^{ai} and R^{bi} . In this section we show how a very particular gauge fixing of this redundancy yields the formula of Hodges [1]. We then prove that the formulas of [1] and [2] are equivalent as a consequence of the matrix tree theorem. Afterwards, we consider reference 3-planes which are fully S_n symmetric. These gauge choices yield new, manifestly S_n symmetric forms of the MHV amplitude.

3.1 Old Representations

The very simplest gauge fixing of L^{ai} and R^{ai} yields the formula of [1]. In particular, we set

$$\begin{aligned} L^{ai} &= \delta^{al_i} \\ R^{ai} &= \delta^{ar_i}, \end{aligned} \quad (22)$$

where l_i and r_i each denote an ordered triplet of external particles. Thus, $\sum_i L^{ai} R^{ai}$ is an $n \times n$ matrix whose entries are all zero except at $\{l_i, r_i\}$ for $i = 1, 2, 3$. Eq. (22) implies that

$$\det \left(\phi^{ab} + \sum_i L^{ai} R^{ai} \right) = \det (\phi_{l_1 l_2 l_3, r_1 r_2 r_3}^{ab}), \quad (23)$$

where $\phi_{l_1 l_2 l_3, r_1 r_2 r_3}^{ab}$ is the $(n-3) \times (n-3)$ reduced matrix obtained by removing the rows l_i and the columns r_i . For simplicity we ignore the overall sign of the amplitude, since it is unimportant and anyway just changes by $\text{sgn}(\{l_1 l_2 l_3 12 \dots l_1 l_2 l_3 \dots n\} \rightarrow \{r_1 r_2 r_3 12 \dots r_1 r_2 r_3 \dots n\})$. Finally, plugging Eq. (22) into Eq. (20) yields

$$\hat{\mathcal{M}}_n^{\text{MHV}} = \frac{\det(\phi_{l_1 l_2 l_3, r_1 r_2 r_3}^{ab})}{\langle l_1 l_2 \rangle \langle l_2 l_3 \rangle \langle l_3 l_1 \rangle \langle r_1 r_2 \rangle \langle r_2 r_3 \rangle \langle r_3 r_1 \rangle}, \quad (24)$$

which is precisely the Hodges form of the MHV gravity amplitude. As proven in [1], any choice for l_i and r_i and Eq. (24) will produce the same final answer. The maximally permutation symmetric choice is of course $l_i = r_i$, which preserves a manifest S_{n-3} symmetry.

The formula of [1] can be easily shown to be equivalent to the manifestly S_{n-2} symmetric expression of [2]. To show this, we rescale the entries of ϕ^{ab} so that the full matrix has uniform weight,

$$\tilde{\phi}^{ab} = \phi^{ab} \times \langle ax \rangle \langle ay \rangle \langle bx \rangle \langle by \rangle, \quad (25)$$

which implies that

$$\tilde{\phi}^{aa} = - \sum_{b \neq a} \tilde{\phi}^{ab}, \quad (26)$$

so the elements of any row or column of $\tilde{\phi}^{ab}$ sum to zero. Note that this relation is an algebraic identity fixed by Eq. (9).

Plugging in $\{x, y\} = \{1, 2\}$ for the reference spinors and setting $l_i = r_i = \{1, 2, 3\}$, we find that Eq. (24) becomes

$$\begin{aligned} \hat{\mathcal{M}}_n^{\text{MHV}} &= \frac{\det(\phi_{123,123}^{ab})}{\langle 12 \rangle^2 \langle 23 \rangle^2 \langle 31 \rangle^2} \\ &= \frac{1}{\langle 12 \rangle^2} \left(\prod_{a>2}^n \frac{1}{\langle a1 \rangle^2 \langle a2 \rangle^2} \right) \det(\tilde{\phi}_{123,123}^{ab}), \end{aligned} \quad (27)$$

where the products of angle brackets arise from the rescaling of the rows and columns of ϕ^{ab} to go to $\tilde{\phi}^{ab}$ variables.

From Eq. (25) it is clear from our choice of reference spinors that the first and second rows and columns of $\tilde{\phi}^{ab}$ vanish identically. Thus we need only consider the reduced $(n-2) \times (n-2)$ matrix, $\tilde{\phi}_{12,12}^{ab}$. By definition, $\det(\tilde{\phi}_{123,123}^{ab})$ is a cofactor of the matrix $\tilde{\phi}_{12,12}^{ab}$ with the third row and column removed. Since $\tilde{\phi}_{12,12}^{ab}$ is by construction S_{n-2} symmetric on its remaining indices, this implies that all cofactors are equal, so

$$\det(\tilde{\phi}_{123,123}^{ab}) = \det(\tilde{\phi}_{12l_3,12l_3}^{ab}), \quad (28)$$

for $l_3 \in \{3, \dots, n\}$. Any matrix whose rows and columns sum to zero possesses a null eigenvector whose components are all equal. This eigenvector is invariant under S_n transformations. As is well-known, for such a matrix all cofactors are the same and equal to

$$\det(\tilde{\phi}_{12l_3,12l_3}^{ab}) = (n-2)^{-1} \text{pdet}(\tilde{\phi}_{12,12}^{ab}), \quad (29)$$

where pdet is the product of all eigenvalues modulo for the zero corresponding to the S_n invariant null eigenvector. The $(n-2)$ symmetry factor in Eq. (29) arises from the fact that the index a in Eq. (28) can take on that many values.

The indices of $\tilde{\phi}_{12,12}^{ab}$ correspond to the external points $\{3, 4, \dots, n\}$. As in [2], we can identify these legs with the vertices of a graph, and associate with each pair $\{a, b\}$ a corresponding link. With this mapping, Eq. (29) is recognized as a formulation of the matrix tree theorem, which says that

$$(\# \text{ of trees}) = (\# \text{ of vertices})^{-1} \text{pdet}(\text{Laplacian matrix}), \quad (30)$$

where the Laplacian matrix is defined as the difference between the degree matrix and the adjacency matrix. Like $\tilde{\phi}^{ab}$, the Laplacian matrix has rows and columns which sum to zero. Hence, Eq. (29) is a “weighted” version of Eq. (30) in which each link of the tree is accompanied by the corresponding factor from $\tilde{\phi}^{ab}$. This yields the expression

$$\hat{\mathcal{M}}_n^{\text{MHV}} = \frac{1}{\langle 12 \rangle^2} \left(\prod_{a>2}^n \frac{1}{\langle a1 \rangle^2 \langle a2 \rangle^2} \right) \sum_{\text{trees}} \prod_{\text{edges } ab} \frac{[ab]}{\langle ab \rangle} \times \langle a1 \rangle \langle a2 \rangle \langle b1 \rangle \langle b2 \rangle, \quad (31)$$

where the summation is over all possible trees which span a labeled graph with $(n-2)$ vertices. This proves the equivalence of the MHV formulae in [1] and [2].

3.2 New Representations

One can go beyond the formulas in the existing literature by opting for more exotic gauge fixings. Most gauge fixings of L^{ai} and R^{ai} will explicitly break the S_n symmetry of the amplitude, since these are 3-planes residing in an n -dimensional space. However, there is a S_n covariant 3-plane which can be chosen as the reference, namely

$$L^{ai} = R^{ai} = v_a^i. \quad (32)$$

Note, crucially, the difference in the raised and lowered operators on the left and right hand sides. While this gauge fixing is S_n covariant, it explicitly breaks the full $GL(n, \mathbb{C})$. However, because this $GL(n, \mathbb{C})$ breaking enters precisely through a gauge redundancy, the final answer is still $GL(n, \mathbb{C})$ invariant, as required by Eq. (17).

With this gauge fixing, we obtain a new, manifestly S_n symmetric form of the MHV gravity amplitude

$$\begin{aligned}\hat{\mathcal{M}}_n^{\text{MHV}} &= \left(\sum_{a,b,c} v_{abc}^2 / 3 \right)^{-2} \det(\phi^{ab} + v_{ab}) \\ &= \left(\sum_{a,b,c} \frac{\langle ab \rangle^2 \langle bc \rangle^2 \langle ca \rangle^2}{12} \right)^{-2} \det\left(\phi^{ab} + \frac{\langle ab \rangle^2}{2}\right)\end{aligned}\quad (33)$$

Let us comment on some remarkable features of this formula. As noted earlier, this gauge fixing explicitly breaks $GL(n, \mathbb{C})$. Since the little group is a subgroup of the broken $GL(n, \mathbb{C})$, this expression is not manifestly homogenous under spinorial weights. This is clear, for example, from the denominator factor, which is a sum over monomials of different spinor weight. Furthermore, the expression is not manifestly homogenous in mass dimension since ϕ^{ab} is a dimensionless phase while v_{ab} has mass dimension 2. Of course, Eq. (33) still gives the final correct answer, which has both correct spinorial weight and correct mass dimension. This was guaranteed by the underlying gauge redundancy in the MHV amplitude.

There is another S_n symmetric choice of reference 3-planes which can be made using the anti-holomorphic spinor $\tilde{\lambda}^a$. In particular, we choose

$$L^{ai} = R^{ai} = \tilde{v}^{ai} \quad (34)$$

where we have defined the anti-holomorphic 3-plane in the obvious way,

$$\tilde{v}^{ai} = \frac{i}{2} \sum_{\alpha, \beta} \tilde{\lambda}^{a\dot{\alpha}} \bar{\sigma}^i_{\dot{\alpha}\beta} \tilde{\lambda}^{a\dot{\beta}}. \quad (35)$$

This yields yet another S_n symmetric form of the amplitude,

$$\hat{\mathcal{M}}_n^{\text{MHV}} = \left(\sum_{a,b,c} \frac{s_{ab}s_{bc}s_{ca}}{12} \right)^{-2} \det\left(\phi^{ab} + \frac{[ab]^2}{2}\right). \quad (36)$$

While the above expression is clearly little group covariant, it is inhomogeneous in mass dimensions. Again, this does not matter because of the intrinsic gauge redundancy of the MHV amplitude.

4 Future Directions

This paper leaves numerous possible directions for future work. The leading open question is conjectural: might $GL(n, \mathbb{C})$ be a hidden symmetry of all gravity amplitudes? To evaluate this

possibility, an understanding of the space of geometric constraints relevant to the tree level N^{k-2} MHV amplitudes will be essential. Given the substantial evidence for a hidden $GL(n, \mathbb{C})$ at the MHV level, it may even be that higher loop amplitudes can be similarly constructed.

The key players in this paper were the auxiliary variables ϕ^{ab} and v_a^i . We have seen hints that these quantities are, fundamentally, integration variables which have been localized to their values in Eq. (3). This is certainly suggested by the transformation law in Eq. (18), which would arise from an integration measure over $GL(n, \mathbb{C})$ covariant auxiliary variables. In this way, gravity amplitudes could nicely mirror the Grassmannian twistor formulation of $\mathcal{N} = 4$ SYM [7], which has been discussed in great depth in numerous papers [6, 7, 8, 9], among others. There is also a likely connection between our results and the recently proposed formulas for $\mathcal{N} = 8$ gravity [10, 11], which have already been studied in some recent work [12, 13, 14].

Note added: During the completion of this work [4] and [5] also pointed out the equivalence of the MHV amplitudes in [1] and [2] using the matrix tree theorem.

Acknowledgements

C. C. is supported by the Director, Office of Science, Office of High Energy and Nuclear Physics, of the US Department of Energy under Contract DE-AC02-05CH11231, and by the National Science Foundation under grant PHY-0855653. C.C. is indebted to Nima Arkani-Hamed for a timely reminder of C.C.'s earlier unpublished note from 2009 relating MHV amplitudes to the matrix tree theorem.

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